n-Coloring Euclidean Homogeneous Spaces

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Abstract

Many geometrically interesting spaces can be derived from n-space as the quotient space resulting from a discrete subgroup of the group of isometries of R^n. The most familiar of these are the torus, the infinite tube, and the cone, all of various dimensions. The projection of R^n onto such a homogeneous space, call it M, defines a metric on M in which the images of lines are geodesics. The images of planes are pseudo-flat structures that we will refer to as pseudo-planes. A coloring of any such homogeneous space lifts to a coloring of R^n by taking the entire coset of a point x in M to by painted by the color assigned to x. This paper addresses questions concerning the coloring of pseudo-flat planes, and geodesics in particular, in M, by applying corresponding findings for coloring of n-space.

Key Words: coloring, homogeneous space, torus, cone

Introduction

The torus of dimension n is derived from R^n as the quotient group of R^n by the integer lattice in n-space, a discrete set that can be characterized as the orbit of the origin under a group of translations by (p,q) for integers p and q, which is a group of isometries of R^n. When using the subgroup characterization, it is necessary to insure not only that the subgroup is not only generated by discrete isometries, but that the whole subgroup they generate is still discrete. The subgroup generated by a general translation and a general rotation is not usually discrete, but the following propositions are in any case true for discrete isometry groups. The infinite tube is the homogeneous space defined by the group of translations along only one axis. The cone is a homogeneous space obtained as the quotient of n-space by a discrete rotation group of finite order, e.g. rotation through $2\pi/m$. Other, more complicated subgroups can be derived in natural ways from these examples, but it is not so easy to characterize the resulting spaces in words. In any case, let M denote any such homogeneous space, of dimension n, and let Γ denote an (n+2)-coloring of M. We prove

Theorem: For every whole number $p \geq 1$ there is a pseudo-plane in M (for p=1, it is a geodesic) which meets p+2 colors.

This theorem has an alternate statement which more clearly expresses it relationship to Euclidean Ramsey Theory. In R^n, for any subgroup H of the non-singular affine group, finite subset X, and natural number m, a triple (H,X,m) is said to have the Ramsey property if for every m-coloring of R^n the orbit of X under H is rich enough to contain a monochromatic set. In analogous fashion, we define a condition called incidence.

Def: For any m-coloring Γ of R^n, subgroup H of the non-singular affine group, subset X of R^n, define the incidence of the five-tuple (Γ,H,X,m,n) to be the maximum number of colors that occur on any set h(X), h in H, in the orbit of X under H. Define the incidence of the four-tuple (H,X,m,n) to be the infimum of (Γ,H,X,m), for all colorings Γ.

This definition can easily be extended to the class of quotient spaces defined above. All that is necessary to obtain a consistent coloring of M is to restrict the class of colorings of R^n to those for which the cosets of every point in M are monochromatic. In these terms, the Theorem is equivalent to

Corollary: For H to be the entire non-singular affine group of R^n, and for X a hyper-plane of dimension k, the incidence of (H,X,m,n) equals k+2.

Preliminaries.

The following proposition seems to be well known, but we have not been able to identify a single, tractable reference for it. Accordingly, we will demonstrate it here for the sake of completeness.

Lemma: Let Γ be a (n+2)-coloring of R^n. for every p=1,2,…,(n-1), there is a p-dimensional hyperplane X which contains p+2 colors.

Pf. Given any such coloring, define an n+3 coloring of the n-sphere, S^n, by pulling R^n back to S^n by stereographic projection, and by then coloring the North Pole by the n+3rd color. Two cases arise, depending on whether n=2 or n>2.

If n=2, stereographic projection results in a 5-colored 2-sphere. We know from [Gibbons] that there is a 4-colored circle. If that circle passes through the North Pole, we are done. If not, let Γ_i, i=1,2,3,4 by the four colors on circle C, and chose x_i in C∩ Γ_i. Number these sets so that the line [x_1x_2] separates x_3 and x_4. Lines [x_1x_2] and [x_3x_4] therefore cross at x, inside C. No matter the color of x, it represents a third color on one or the other of these lines.

If n>2, from [Gibbons] we know that there is an (n-1)-sphere in S^n containing n+2 colors. If that sphere passes through the Noth Pole, it projects to a hyperplane containing n+1 colors in R^n, and we are done. Otherwise, the (n+2)-colored sphere projects to a sphere S in R^n containing n+2 colors. That sphere contains, by the same theorem, an (n-2)-sphere that contains n+1 colors. Call that sphere $S^{(n-2)}$. It is the locus of intersection of the (n-1)-sphere with a hyperplane P of dimension n-1. Therefore, $S^{(n-2)}$ is a subset of P containing n+1 colors. Thus, there must be a hyperplane containing n+1 colors. This plane, as a copy of $R^{(n-1)}$, contains an (n-2)-dimensional hyperplane which contains n colors, and so on, counting backwards to a line that must contain three colors. Q.e.d.

Proof of the Theorem and Corollary

Pf. of the Theorem. Given an (n+2)-coloring Γ of an n-dimensional homogeneous space M, construct an (n+2)-coloring of R^n by imposing the condition that the coset in R^n of any point x in M is monochromatic. For every p there is a hyperplane which contains p+2 colors, for p=1,2,…,n-1. The quotient of this plane in M satisfies the condition asserted in the theorem. In particular, there is a four-colored geodesic. Q.e.d.

Pf. of the Corollary. For H the affine group of R^n and X a hyperplane in R^n, the orbit of X under H is simply the set of all hyperplanes of the same dimension as X. The inf-sup condition implies that for every coloring of M, there is a hyperplane that is at least (p+2)-colored, where p is the dimension of X. This however simply restates the conclusion of Theorem 1.

Examples

The class of 2-dimensional homogeneous spaces affords a rich collection of specific examples of this theorem.

1. The 2-torus. The geodesics on T^2 consist of quotients of lines in the plane. Lines that pass through two points in the integer lattice project to closed loops. Those that do not – those having irrational slope – project to lines that wind densely around the torus. The difference between these cases brings home the fact that in reference to the second statement of the theorem, the group H is defined on Euclidean space, and may not be defined on M.
2. The infinite tube. Geodesics on the tube are again of two types. One type consists of cross sections of the tube. The other type consists of helices extending infinitely in both directions.

3. The cone. Cones are obtained from R^2 by quotienting with respect to a discrete subgroup of the rotation group of the plane. The geodesics are either cross sections of the cone or expanding helices that wind some finite number of times around the cone. In general, the number of complete winds is equal to the integer m defined by the rotation group. The example has an added complication of interest. A given cone can be modeled as the quotient by any discrete rotation group. Thus for instance let G(2) be the group of rotations by π: m=2. A line in the plane is parallel to some line through the origin. The quotient of a line through the origin consists of a pair of raw from the apex – running on opposite side of the cone. The quotient of a line parallel to a line through the origin is an hyperbola asymptotic to the corresponding rays.

Bibliography

[Gibbons] J. Gibbons, "(n+3)-Coloring the n-Sphere." Logistic Research & Trading Co. Working Paper, Sept. 2010.

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